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A CYLINDRICAL SHELL WITH AN AXIAL CRACK UNDER SKEW-SYMMETRIC LOADING†

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Abstract—The skew-symmetric problem for a cylindrical shell containing an axial crack is considered. It is assumed that the material has a special orthotropy, namely that the shear modulus may be evaluated from the measured Young's moduli and the Poisson's ratios rather than being an independent material constant. The problem is solved within the confines of an eighth order linearized shallow shell theory. As numerical examples the torsion of an isotropic cylinder and that of a specially orthotropic cylinder (titanium) are considered. The membrane and bending components of the stress intensity factor are calculated and are given as functions of a dimensionless shell parameter. In the torsion problem for the axially cracked cylinder the bending effects appear to be much more significant than that found for the circumferentially cracked cylindrical shell. Also, as the shell parameter increases, unlike the results found in the pressurized shells, the bending stresses around crack ends do not change sign.

NOTATION

a	half crack length.
$c = (E_1/E_2)^{1/4}$	
$C_m = k_s^m/k_p$	membrane component of stress intensity ratio.
$C_b \quad k_s^b/k_p$	bending component of stress intensity ratio.
$D_k = E_k h^3/12(1-\nu_1\nu_2), (k = 1, 2)$	
$E_1, E_2, \nu_1, \nu_2, G_{12}$	elastic constants of the orthotropic shell.
F	stress function.
h	shell thickness.
$k_p = N_0 a^{1/2}/h$	flat plate stress intensity factor.
k_s^m, k_s^b	membrane and bending components of the stress intensity factor in the shell.
$N_{ij}, M_{ij}, V_i, (i, j = X, Y)$	stress, moment, and effective shear resultants in the shell.
N_0	the external load, uniform shear N_{XY} away from the crack.
w	Z-component of the displacement in the shell.
$x = X/a, y = cY/a$	dimensionless coordinates.
$\sigma_{ij}^m, \sigma_{ij}^b, (i, j = x, y)$	membrane and bending stresses in the shell.
$\lambda = [12(1-\nu_1\nu_2)]^{1/4} \dfrac{a}{c(Rh)^{1/2}}$	dimensionless parameter for specially orthotropic shell.

1. INTRODUCTION

SINCE the discovery of the fact that the curvature may have a significant effect on the stress intensity factors in curved sheets containing a through crack, in recent years there has been a considerable interest in the crack problems of shells (see, for example, [1–9]). The general problem of a cracked shell with arbitrary radii of curvatures does not seem to lend itself to any kind of a tractable analysis. Hence, up to now only the shells with idealized geometries, namely, the shallow cylindrical and spherical isotropic shells containing a

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meridional crack, have been studied. In practical applications it is then understood that the stress intensity factors necessary for the analysis of fracture and, particularly, fatigue crack propagation in a given shell may be obtained from the perturbation problem for a cylindrical or for a spherical shell approximating the actual shell structure in the neighborhood of the crack. For this, it is clear that one needs the solution of the idealized crack problems for cylindrical and spherical shells under both symmetric and anti-symmetric loading conditions. With the exception of [9], where the torsion problem for a cylindrical shell containing a circumferential crack is considered, the previous studies on cracked shells deal entirely with the symmetric loading conditions.

Another problem in shells is the investigation of the effect of material anisotropy on the stress intensity factors. In this case, however, even with the assumption of orthotropy, the related differential operators are not factorable, which again makes the analysis intractable†. On the other hand, if one assumes that the material possesses a property of special orthotropy‡, the related differential operators become factorable, and the characteristic equation can be solved in closed form. In this case the problem can be solved without any difficulty.

The purpose of this paper is then to obtain the solution of the problem of a specially orthotropic cylindrical shell which contains a longitudinal crack and is subjected to torsion. The result for the isotropic shell is obtained as a special case (see [17]).

2. FACTORIZATION OF THE DIFFERENTIAL OPERATORS

The linear bending theory of anisotropic shallow shells dates back to a paper by Ambartsumyan [10]. The detailed treatment of the subject may be found in [11–13]. Here we will simply repeat the relevant equations in an eighth order theory for an orthotropic shallow cylindrical shell. Referring to Fig. 1, let $2a$ be the crack length, h be the thickness and R be the mean radius of curvature in the shell. Assume that the problem for the shell without the crack and subjected to the given set of external loads has been solved, and by proper superposition the problem has been reduced to one in which the membrane and bending loads applied to the crack surface are the only external loads. Referred to

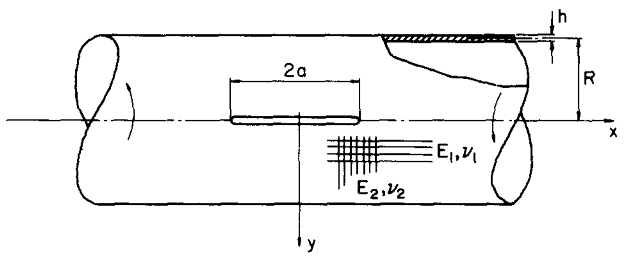

FIG. 1. Cylindrical shell with an axial crack.

† By using the method given in this paper, the orthotropic cylindrical and spherical shell problems can be handled (in principle) by extracting the asymptotic behavior of the roots of the characteristic equation for small and large values of the transform variable, by separating the dominant part of the singular integral equations based on these asymptotic values, and by lumping the effects of the remaining, numerically evaluated parts with the Fredholm kernels. However, this would require extremely involved and lengthy numerical work.

‡ This is an orthotropic sheet in which the shear modulus is not independent of the elastic constants E_1, E_2, ν_1, and ν_2; that is, the material has three rather than four independent elastic constants.

the orthogonal nondimensional coordinates defined by

$$x_1 = X/a, \qquad x_2 = Y/a, \tag{1}$$

the differential equations for an orthotropic shallow cylindrical shell may be expressed as [11, 13],

$$D_1\nabla_1^4 w(x_1, x_2) - \frac{a^2}{R}\frac{\partial^2}{\partial x_1^2}F(x_1, x_2) = 0,$$
$$\nabla_2^4 F(x_1, x_2) + \frac{hE_2 a^2}{R}\frac{\partial^2}{\partial x_1^2}w(x_1, x_2) = 0, \tag{2a, b}$$

where F is a stress function and w is the displacement component normal to the surface. The following expressions for the stress strain relations in an orthotropic sheet define the notation for the elastic constants:

$$\varepsilon_{11} = \frac{1}{E_1}(\sigma_{11} - \nu_1\sigma_{22}),$$
$$\varepsilon_{22} = \frac{1}{E_2}(\sigma_{22} - \nu_2\sigma_{11}), \tag{3a–d}$$
$$\varepsilon_{12} = \frac{1}{2G_{12}}\sigma_{12}, \qquad \frac{\nu_1}{E_1} = \frac{\nu_2}{E_2}.$$

Here 1 and 2, or x_1 and x_2 are the principal directions of orthotropy in the shell. The differential operators ∇_1^4 and ∇_2^4 are defined by

$$\nabla_1^4 = \frac{\partial^4}{\partial x_1^4} + 2\left[\nu_2 + 2(1-\nu_1\nu_2)\frac{G_{12}}{E_1}\right]\frac{\partial^4}{\partial x_1^2\partial x_2^2} + \frac{E^2}{E_1}\frac{\partial^4}{\partial x_2^4},$$
$$\nabla_2^4 = \frac{\partial^4}{\partial x_1^4} + 2\left(\frac{E_2}{2G_{12}} - \nu_2\right)\frac{\partial^4}{\partial x_1^2\partial x_2^2} + \frac{E_2}{E_1}\frac{\partial^4}{\partial x_2^4}. \tag{4a, b}$$

The stress and moment resultants are related to F and w through the following expressions:

$$N_{11} = \frac{1}{a^2}\frac{\partial^2 F}{\partial x_2^2}, \qquad N_{22} = \frac{1}{a^2}\frac{\partial^2 F}{\partial x_1^2}, \qquad N_{12} = -\frac{1}{a^2}\frac{\partial^2 F}{\partial x_1 \partial x_2},$$
$$M_{11} = -\frac{D_1}{a^2}\left(\frac{\partial^2 w}{\partial x_1^2} + \nu_2\frac{\partial^2 w}{\partial x_2^2}\right), \qquad M_{22} = -\frac{D_2}{a^2}\left(\frac{\partial^2 w}{\partial x_2^2} + \nu_1\frac{\partial^2 w}{\partial x_1^2}\right),$$
$$V_1 = Q_1 + \frac{\partial M_{12}}{\partial x_2} = -\frac{D_1}{a^3}\left[\frac{\partial^3 w}{\partial x_1^3} + \left(\nu_2 + \frac{h^3 G_{12}}{3D_1}\right)\frac{\partial^3 w}{\partial x_1 \partial x_2^2}\right], \tag{5a–g}$$
$$V_2 = Q_2 + \frac{\partial M_{12}}{\partial x_1} = -\frac{D_2}{a^3}\left[\frac{\partial^3 w}{\partial x_2^3} + \left(\nu_1 + \frac{h^3 G_{12}}{3D_2}\right)\frac{\partial^3 w}{\partial x_1^2 \partial x_2}\right],$$

where

$$D_k = E_k h^3/12(1-\nu_1\nu_2), \qquad (k = 1, 2). \tag{6}$$

The membrane and bending components of the stresses are obtained from the relations of the form

$$\sigma_{11}^{m} = N_{11}/h, \ldots, \sigma_{11}^{b} = 12M_{11}Z/h^{3}, \ldots \tag{7}$$

By expressing the unknown functions F and w in terms of appropriate Fourier integrals, (2) may be reduced to a system of two fourth order ordinary differential equations. The characteristic function of this system will then be an eighth degree polynomial the coefficients of which will be functions of the transform variable. For the solution of the problem, it is essential that the roots of this polynomial be obtainable in closed form. In the present problem this is possible only if the elastic constants of the material are such that the operators given by (4) can be factorized. From (4), it is clear that these operators can be expressed in the following form

$$\nabla_1^4 = \left[\frac{\partial^2}{\partial x_1^2} + \sqrt{(E_2/E_1)}\cdot\frac{\partial^2}{\partial x_2^2}\right]^2 = \nabla_2^4 \tag{8}$$

provided the elastic constants satisfy the following conditions (see [17]):

$$\left[\nu_2 + 2(1-\nu_1\nu_2)\frac{G_{12}}{E_1}\right]\sqrt{E_1/E_2)} = 1,$$
$$\left(\frac{E^2}{2G_{12}} - \nu_2\right)\sqrt{(E_1/E_2)} = 1. \tag{9a, b}$$

By direct substitution, it can be shown that the conditions (9) will be satisfied if

$$G_{12} = \frac{(E_1E_2)^{\frac{1}{2}}}{2[1+\sqrt{(\nu_1\nu_2)}]}. \tag{10}$$

Considering also the relation $(\nu_1/E_1) = (\nu_2/E_2)$, this means that the material has three independent elastic constants and the fourth constant G_{12} is obtained in terms of an "average" Young's modulus $(E_1E_2)^{\frac{1}{2}}$ and an "average" Poisson's ratio $(\nu_1\nu_2)^{\frac{1}{2}}$ by using the standard expression for shear modulus in isotropic elastic solids. The plate for which the condition (10) is satisfied is said to be specially orthotropic. The analysis given in this paper will then be valid only for those materials in which the measured value of G_{12} and that calculated from (10) in terms of measured E_i and ν_i, $(i = 1, 2)$ are in reasonably good agreement.

Changing the variables once more as

$$x_1 = x, \qquad (E_1/E_2)^{\frac{1}{4}}x_2 = y, \tag{11}$$

the operators ∇_1^4 and ∇_2^4 reduce to the following biharmonic operators:

$$\nabla_1^4 = \nabla_2^4 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = \nabla^4. \tag{12}$$

With (12), the system of equations (2) becomes identical to the differential equations for isotropic shells in which $D = Eh^3/12(1-\nu^2)$ and E are replaced by D_1 and E_2, respectively.

3. DERIVATION OF THE INTEGRAL EQUATIONS

By using (12), in terms of the new coordinates

$$x = X/a, \qquad y = cY/a, \qquad c = (E_1/E_2)^{\frac{1}{4}}, \tag{13}$$

the differential equations (2) may be expressed as

$$D_1\nabla^4 w(x, y) - \frac{a^2}{R}\frac{\partial^2}{\partial x^2}F(x, y) = 0,$$
$$\nabla^4 F(x, y) + \frac{hE_2 a^2}{R}\frac{\partial^2}{\partial x^2}w(x, y) = 0. \tag{14a, b}$$

For the stress perturbation problem, (14) must be solved under the following external loads applied to the crack surface:

$$\lim_{Y\to\mp 0} M_Y(X, Y) = -\lim_{y\to\mp 0}\frac{D_2}{a^2}\left(c^2\frac{\partial^2 w}{\partial y^2} + \nu_1\frac{\partial^2 w}{\partial x^2}\right) = 0,$$
$$\lim_{Y\to\mp 0} N_Y(X, Y) = \lim_{y\to\mp 0}\frac{1}{a^2}\frac{\partial^2 F}{\partial x^2} = 0,$$
$$\lim_{Y\to\mp 0} N_{XY}(X, Y) = -\lim_{y\to\mp 0}\frac{c}{a^2}\frac{\partial^2 F}{\partial x\partial y} = -N^0_{XY}(X) = -t_0(x), \tag{15a–d}$$
$$\lim_{Y\to\mp 0} V_Y(X, Y) = -\lim_{y\to\mp 0}\frac{D_2}{a^3}\left[c^3\frac{\partial^3 w}{\partial y^3} + c\left(\nu_1 + \frac{h^3 G_{12}}{3D_2}\right)\frac{\partial^3 w}{\partial^2 x\partial y}\right]$$
$$= -V^0_Y(X) = -v_0(x), \qquad (-1 < x < 1),$$

where $t_0(x) = t_0(-x)$ and $v_0(x) = -v_0(-x)$. Also, outside the crack, the antisymmetry of the problem and the conditions of continuity require that

$$M_Y(X, 0) = 0, \qquad N_Y(X, 0) = 0,$$
$$\lim_{y\to +0}\frac{\partial^n}{\partial y^n}w(x, y) = \lim_{y\to -0}\frac{\partial^n}{\partial y^n}w(x, y), \qquad (n = 0, 1, 2, 3), \tag{16a–d}$$
$$\lim_{y\to +0}\frac{\partial^n}{\partial y^n}F(x, y) = \lim_{y\to -0}\frac{\partial^n}{\partial y^n}F(x, y), \qquad (n = 0, 1, 2, 3), \qquad (|x| > 1).$$

In the problem under consideration, the external loads are self-equilibrating local tractions. Hence the functions w and F satisfy the regularity conditions at $x = \mp\infty$ and, consequently may be expressed in terms of Fourier integrals. Thus, using the Fourier transforms to solve (14), and taking into consideration the antisymmetric nature of the problem, after routine manipulations we obtain

$$w(x, y) = \mathrm{sgn}(y)\int_0^\infty \sum_1^4 Q_j(\alpha)\,\mathrm{e}^{m_j|y|}\sin\alpha x\,\mathrm{d}\alpha,$$
$$F(x, y) = \mathrm{sgn}(y)\int_0^\infty \sum_1^4 K_j Q_j(\alpha)\,\mathrm{e}^{m_j|y|}\sin\alpha x\,\mathrm{d}\alpha, \tag{17a, b}$$

where

$$
\begin{aligned}
K_1 &= K_2 = -i(E_2hD_1)^{1/2}, & K_3 &= K_4 = i(E_2hD_1)^{\frac{1}{2}},\\
m_1 &= -(\alpha^2+i_1\lambda\alpha)^{1/2}, & m_2 &= -(\alpha^2-i_1\lambda\alpha)^{\frac{1}{2}},\\
m_3 &= -(\alpha^2+i_2\lambda\alpha)^{1/2}, & m_4 &= -(\alpha^2-i_2\lambda\alpha)^{\frac{1}{2}},\\
i_1 &= e^{\pi i/4}, \quad i_2 = e^{-\pi i/4}, & \lambda^4 &= 12(E_2/E_1)(1-\nu_1\nu_2)\frac{a^4}{R^2h^2},
\end{aligned}
\tag{18}
$$

and the functions $Q_j(\alpha)$, $(j = 1,\dots 4)$ are unknown. Combining (15a, b) and (16a, b), and substituting from (17), we obtain the following two algebraic equations for Q_j:

$$
\begin{aligned}
Q_3 &= \left(\frac{\alpha(\nu_1-c^2)}{i_2\lambda c^2}+\frac{1}{2}\right)(Q_1+Q_2)-\frac{i}{2}(Q_1-Q_2),\\
Q_4 &= -\left(\frac{\alpha(\nu_1-c^2)}{i_2\lambda c^2}-\frac{1}{2}\right)(Q_1+Q_2)+\frac{i}{2}(Q_1-Q_2).
\end{aligned}
\tag{19a, b}
$$

By considering the mixed boundary conditions (15c, d) and (16c, d) at $y = 0$, two more relations for Q_j may be obtained in the form of a system of dual integral equations. Since, by definition w and F are odd functions, in (16c, d) the conditions for $n = 1$ and $n = 3$ are automatically satisfied, and analytically (16c, d) simply means that the functions which are odd in y must vanish for $y = 0$, $|x| > 1$. Using (19), after some algebraic manipulations, these conditions may be stated as

$$
\begin{aligned}
&\int_0^\infty (Q_1+Q_2)\sin\alpha x\,d\alpha = 0\\
&\int_0^\infty (Q_1+Q_2)\alpha^2\sin\alpha x\,d\alpha = 0\\
&\int_0^\infty \frac{\nu_1-c^2}{c^2}(Q_1+Q_2)\alpha^2\sin\alpha x\,d\alpha+\int_0^\infty i_1\lambda(Q_1-Q_2)\alpha\sin\alpha x\,d\alpha = 0, \quad (|x|>1).
\end{aligned}
\tag{20a–c}
$$

Here (20a) and (20b) are, respectively, the statement of the conditions that w and $\partial^2 w/\partial y^2$ vanish on $y = 0$, $|x| > 1$. Analytically, since (20b) follows from (20a), (20) is equivalent to only two independent conditions. Noting also that because of (20b) the first integral in (20c) is zero, we select these two conditions as follows:

$$
\begin{aligned}
&\int_0^\infty (Q_1+Q_2)\alpha^2\sin\alpha x\,d\alpha = 0,\\
&\int_0^\infty i_1\lambda(Q_1-Q_2)\alpha\sin\alpha x\,d\alpha = 0, \quad (|x|>1).
\end{aligned}
\tag{21a, b}
$$

Here, (20b) instead of (20a) is selected because of dimensional consistency (with (20c) or (21b)) and we note in passing that the condition $w = 0$ for $y = 0$, $|x| > 1$, [i.e. (20a)] still remains to be satisfied.

Substituting now from (17) into (15c, d) and, again for dimensional consistency, integrating (15d) once, we obtain

$$\lim_{y\to+0}\left[-\frac{c}{a^2}\int_0^\infty \sum_1^4 K_j m_j Q_j\, e^{m_j y}\alpha\cos\alpha x\,d\alpha\right] = -t_0(x),$$

$$\lim_{y\to+0}\int_0^x\left\{-\frac{D_2}{a^3}\int_0^\infty\sum_1^4\left[c^3m_j^3-\alpha^2cm_j\left(v_1+\frac{h^3G_{12}}{3D_2}\right)\right]Q_j\,e^{m_jy}\sin\alpha x\,d\alpha\right\}dx \quad (22a, b)$$

$$= -\int_0^x v_0(x)\,dx, \qquad (|x|<1).$$

With (19), (21) and (22) give the system of dual integral equations to determine the unknown functions Q_1 and Q_2.

The dual integral equations (21) and (22) will be solved by reducing them to a system of singular integral equations. For this purpose we define the following auxiliary functions:

$$u_1(x) = \int_0^\infty i_1\lambda\alpha(Q_1-Q_2)\sin\alpha x\,d\alpha,$$

$$u_2(x) = \int_0^\infty \alpha^2(Q_1+Q_2)\sin\alpha x\,d\alpha, \qquad (0\le x\le\infty). \quad (23a, b)$$

Note that physically u_1 and u_2 are related to the second derivatives of w and F, and hence, have the same type of singularity as N_{ij} and M_{ij} at the crack tips ($x = \mp 1, y = 0$). Inverting (23) and using (21) we find

$$Q_1(\alpha)-Q_2(\alpha) = \frac{2}{\pi i_1\lambda\alpha}\int_0^1 u_1(t)\sin\alpha t\,dt,$$

$$Q_1(\alpha)+Q_2(\alpha) = \frac{2}{\pi\alpha^2}\int_0^1 u_2(t)\sin\alpha t\,dt. \quad (24a, b)$$

Substituting now from (24) into (22) and using (19), we obtain the system of integral equations to determine the new unknown functions u_1 and u_2. If we take into consideration the symmetry properties $u_1(x) = -u_1(-x)$, $u_2(x) = -u_2(-x)$, these integral equations may be expressed as

$$\lim_{y\to+0}\frac{1}{\pi}\int_{-1}^1\sum_1^2 h_{ij}(x,t,y)u_j(t)\,dt = f_i(x), \qquad (|x|<1), \quad (25)$$

where the kernels h_{ij}, $(i, j = 1, 2)$, are given in Appendix A and

$$f_1(x) = \frac{ia^2t_0(x)}{c(E_2hD_1)^{\frac{1}{2}}}, \qquad f_2(x) = \frac{a^3}{D_2}\int_0^x v_0(x)\,dx. \quad (26a, b)$$

From Appendix A it is seen that h_{ij} contains infinite integrals of the form

$$\int_0^\infty \alpha^k n_r(\alpha, y)\sin\alpha p\,d\alpha, \qquad (r = 1, 3;\, k = 0, 2)$$

$$\int_0^\infty \alpha^q n_s(\alpha, y)\sin\alpha p\,d\alpha, \qquad (s = 2, 4;\, q = -1, 1), \quad (27a, b)$$

where $p = t-x$, or $p = t$. By studying the behavior of the integrands for small and large values of α, it can be shown that in (27) the integrals corresponding to $(r = 1,3; k = 0)$ and $(s = 2,4; q = -1)$ are uniformly convergent. Hence, in the related terms in (25) the limit can be put under the integral sign and the corresponding kernels are of the Fredholm type. On the other hand, it is not difficult to show that for $y \to 0$, $p \to 0$ the integrals in (27) which correspond to $(r = 1,3;\ k = 2)$ and $(s = 2,4;\ q = 1)$ become divergent. These singular parts of the kernels may easily be separated by examining the asymptotic behavior of the integrands for large values of α. For example, noting that for large α

$$\alpha^2 n_1(\alpha, y) = i_1\lambda\, e^{-\alpha y} + e^{-\alpha y} 0(\alpha^{-2}), \qquad (y > 0) \tag{28}$$

the integral in (27) for $r = 1$, $k = 2$ may be expressed as

$$\int_0^\infty \alpha^2 n_1(\alpha, y) \sin \alpha p\, d\alpha = i_1\lambda \int_0^\infty e^{-\alpha y} \sin \alpha p\, d\alpha + \int_0^\infty [\alpha^2 n_1(\alpha, y) - i_1\lambda e^{-\alpha y}] \sin \alpha p\, d\alpha$$

$$= i_1\lambda \frac{p}{p^2+y^2} + \int_0^\infty \left[\left(\frac{e^{m_1 y}}{m_1} - \frac{e^{m_2 y}}{m_2}\right)\alpha^2 - i_1\lambda\, e^{-\alpha y}\right] \sin \alpha p\, d\alpha \tag{29}$$

where for all values of p and $y \to 0$ the second integral is uniformly convergent. Hence, in (25) the limit can again be put under the integral sign giving a Fredholm-type kernel. The integrated term in (29), substituted into (25), gives a Cauchy kernel. Similarly, for $s = 2$, $q = 1$ we find

$$\int_0^\infty \alpha n_2(\alpha, y) \sin \alpha p\, d\alpha = \frac{2p}{p^2+y^2} + \int_0^\infty \left[\left(\frac{e^{m_1 y}}{m_1} + \frac{e^{m_2 y}}{m_2}\right)\alpha - 2\, e^{-\alpha y}\right] \sin \alpha p\, d\alpha. \tag{30}$$

Thus, after separating the singular parts of h_{ij} and going to limit, (25) becomes

$$\int_{-1}^{1} \sum_1^2 a_{ij} u_j(t) \frac{dt}{t-x} + \int_{-1}^{1} \sum_1^2 k_{ij}(x, t) u_j(t)\, dt = \pi f_i(x) \qquad (|x| < 1), \tag{31}$$

where

$$a_{11} = 1, \qquad a_{12} = 1 - \frac{v_1}{c^2}, \qquad a_{21} = 0,$$
$$a_{22} = -3v_1 + c^2 + \left(1 + \frac{v_1}{c^2}\right)\left(v_1 + \frac{h^3 G_{12}}{3D_2}\right), \tag{32}$$

and the Fredholm kernels $k_{ij}(x, t)$ are given by

$$k_{1j}(x, t) = \int_0^\infty [F_{1j}(\alpha, 0) - a_{1j}] \sin \alpha(t-x)\, d\alpha, \qquad (j = 1, 2),$$

$$k_{21}(x, t) = \int_0^\infty F_{21}(\alpha, 0)[\sin \alpha(t-x) - \sin \alpha t]\, d\alpha,$$

$$k_{22}(x, t) = -\frac{a_{22}}{t} + \int_0^\infty [F_{22}(\alpha, 0) - a_{22}][\sin \alpha(t-x) - \sin \alpha t]\, d\alpha, \tag{33}$$

where the functions $F_{ij}(\alpha, y)$ are given in Appendix A.

Noting that $u_i(x)$, $(i = 1, 2)$ are related to the second derivatives of w and F and referring to (14), we conclude that the elements of the fundamental matrix of the system of singular integral equations (31) are

$$w_1(x) = w_2(x) = (1-x^2)^{-\frac{1}{2}} \tag{34}$$

and the index of the system is $\kappa = 1$. Thus the solution of (31) will involve two arbitrary constants. These constants are determined by using the condition that $w(x, 0) = 0$ for $|x| > 1$, which has not yet been satisfied. From (17), (21), and (23), recalling that $-2u_2(x) = (\partial^2/\partial x^2)w(x, 0)$ and $u_2(x) = 0$ for $|x| > 1$, the continuity condition for w may be expressed as

$$\int_{-1}^{1} u_2(t)\,\mathrm{d}t = 0, \qquad \int_{-1}^{1} \mathrm{d}x \int_{-1}^{x} u_2(t)\,\mathrm{d}t = 0. \tag{35a, b}$$

4. SOLUTION OF THE INTEGRAL EQUATIONS

To solve the system of singular integral equations (31) the technique described in [15] is used. Noting that the fundamental functions of the system as given by (34) are the weight functions of Chebyshev polynomials $T_n(x)$, and $u_1(x) = -u_1(-x)$, $u_2(x) = -u_2(-x)$, the unknown functions may be expressed as

$$u_1(x) = (1-x^2)^{-\frac{1}{2}} \sum_1^{\infty} A_n T_{2n-1}(x),$$
$$u_2(x) = (1-x^2)^{-\frac{1}{2}} \sum_1^{\infty} B_n T_{2n-1}(x), \qquad (|x| < 1) \tag{36a, b}$$

where A_n and B_n are unknown (complex) constants. The condition (35a) is satisfied by the choice of u_2 as in (36b). (35b) gives the same result as that obtained below by directly writing $w = 0$ at $x = \mp 1$, $y = 0$:

$$0 = w(0, 1) = -w(0, -1) = 2\int_0^{\infty} (Q_1 + Q_2)\sin\alpha\,\mathrm{d}\alpha$$
$$= 2\int_0^{\infty} \sin\alpha\,\mathrm{d}\alpha \frac{2}{\pi\alpha^2}\int_0^1 \sum_1^{\infty} B_n T_{2n-1}(x)\frac{\sin\alpha x}{(1-x^2)^{\frac{1}{2}}}\,\mathrm{d}x$$
$$= \frac{4}{\pi}\sum_1^{\infty} B_n \int_0^1 T_{2n-1}(x)\frac{x\,\mathrm{d}x}{(1-x^2)^{\frac{1}{2}}}$$
$$= \frac{2}{\pi}\sum_1^{\infty} B_n \int_{-1}^1 T_{2n-1}(x)T_1(x)\frac{\mathrm{d}x}{(1-x^2)^{\frac{1}{2}}} = B_1. \tag{37}$$

The remaining constants A_n, $(n = 1, 2, \ldots)$ and B_n, $(n = 2, 3, \ldots)$ are determined by substituting from (36) into (31) and following the procedure outlined in [15].

After determining u_1 and u_2, from (23) and (36) we find

$$i_1\lambda\alpha(Q_1 - Q_2) = \sum_1^{\infty} (-1)^{n-1} A_n J_{2n-1}(\alpha),$$
$$\alpha^2(Q_1 + Q_2) = \sum_1^{\infty} (-1)^{n-1} B_n J_{2n-1}(\alpha). \tag{38a, b}$$

With (19) and (17), this formally completes the solution of the problem. In particular, the membrane and bending stresses may be expressed in terms of a series of infinite integrals. For example, for the membrane stress of primary interest, σ_{xy}^{m} we have

$$\sigma_{xy}^{m} = -\frac{c}{ha^2}\frac{\partial^2 F}{\partial x \partial y} = \frac{c}{ha^2}\int_0^{\infty}\sum_1^4 K_j Q_j(\alpha) m_j \alpha\, e^{m_j y}\cos \alpha x\, d\alpha, \qquad (y \geq 0). \tag{39}$$

It can be shown that at ($y = 0$, $x = \mp 1$), the integrals in (39) are divergent, meaning that the stresses will have a singularity at the crack tips. Noting that the integrand in (39) is integrable around $\alpha = 0$ and is bounded and continuous elsewhere in the domain, the divergent behavior of the integral must be due to the asymptotic behavior of the integrand for large values of α. Thus, substituting from (38) and (19), (39) may be expressed as

$$\sigma_{xy}^{m} = \frac{c}{ha^2} i(E_2 h D_1)^{\frac{1}{2}} \sum_1^{\infty}(-1)^{n-1}\left(A_n - \frac{\nu_1 - c^2}{c^2} B_n\right)$$

$$\times \int_0^{\infty} J_{2n-1}(\alpha)[-1 + \alpha y + 0(\alpha^{-1})]\, e^{-\alpha y}\cos \alpha x\, d\alpha \qquad (y \geq 0). \tag{40}$$

Noting that for large values of α [16]

$$J_{2n-1}(\alpha) \cong (-1)^{n-1} J_1(\alpha) \cong \left(\frac{2}{\pi\alpha}\right)^{\frac{1}{2}}(-1)^{n-1}\left[\cos\left(\alpha - \frac{3\pi}{4}\right) + \dots\right] \qquad (n = 1, 2, \dots), \tag{41}$$

and using the results in [16] to evaluate the integrals, we obtain the leading term in the stress expression as follows:

$$\sigma_{xy}^{m}(r, \theta) = \frac{ci}{ha^2}(E_2 h D_1)^{\frac{1}{2}} \sum_1^{\infty}\left(-A_n + \frac{\nu_1 - c^2}{c^2} B_n\right)$$

$$\times \frac{1}{4\sqrt{(2r)}}\left(3\cos\frac{\theta}{2} + \cos\frac{5\theta}{2}\right) + O(r^{\frac{1}{2}}), \tag{42}$$

where (r, θ) are the polar coordinates measured from the crack tip,

$$(x = 1,\ y = 0), \qquad r^2 = (x-1)^2 + y^2, \qquad \tan\theta = y/(-1 + x).$$

As an example, consider the external loads

$$t_0(x) = N_0, \qquad v_0(x) = 0. \tag{43}$$

Defining the following normalized functions [see (26)]:

$$u_j^*(x) = u_j(x)/u_0, \qquad (j = 1, 2),$$

$$u_0 = \frac{a^2 N_0}{c\sqrt{(E_2 h D_1)}} = \frac{\lambda^2 R N_0 c}{h\sqrt{(E_1 E_2)}},$$

$$u_1^*(x) = (1 - x^2)^{-\frac{1}{2}} \sum_1^{\infty} a_n T_{2n-1}(x),$$

$$u_2^*(x) = (1 - x^2)^{-\frac{1}{2}} \sum_1^{\infty} b_n T_{2n-1}(x), \tag{44a–d}$$

(42) may be expressed as

$$\sigma_{xy}^m = \left(\frac{N_0\sqrt{a}}{h}\right)\left[i\sum_1^\infty\left(-a_n+\frac{v_1-c^2}{c^2}b_n\right)\right]\frac{1}{4\sqrt{(2ra)}}\left(3\cos\frac{\theta}{2}+\cos\frac{5\theta}{2}\right)+O(r^{\frac{1}{2}}). \tag{45}$$

Now observing that the stress intensity factor in a flat plate under uniform shear stress N_0/h and that in a shell are defined by

$$k_p = N_0\sqrt{a}/h, \qquad k_s^m = \lim_{r\to 0}(\sqrt{2ra})\sigma_{xy}^m(r,0), \tag{46a, b}$$

from (45) we obtain the membrane component of the stress intensity ratio for the shell as follows (see [17]):

$$C_m = \frac{k_s^m}{k_p} = i\sum_1^\infty\left(-a_n+\frac{v_1-c^2}{c^2}b_n\right). \tag{47}$$

The remaining membrane stresses may be obtained in a similar way. Thus, for small values of r the membrane stresses in the shell may be expressed as

$$\sigma_{xx}^m(r,\theta) = \frac{C_m k_p}{4c\sqrt{(2ra)}}\left(7\sin\frac{\theta}{2}+\sin\frac{5\theta}{2}\right)+O(r^{1/2}),$$

$$\sigma_{yy}^m(r,\theta) = \frac{C_m k_p}{4\sqrt{(2ra)}}\left(-\sin\frac{\theta}{2}+\sin\frac{5\theta}{2}\right)+O(r^{\frac{1}{2}}), \tag{48a–c}$$

$$\sigma_{xy}^m(r,\theta) = \frac{C_m k_p}{4\sqrt{2ra)}}\left(3\cos\frac{\theta}{2}+\cos\frac{5\theta}{2}\right)+O(r^{\frac{1}{2}}).$$

Also, defining the bending component of the stress intensity factor by

$$k_s^b = \lim_{r\to 0}\sqrt{(2ra)}\sigma_{xy}^b(r,0) = C_b k_p, \tag{49}$$

in a similar way the asymptotic expressions for the bending stresses around the crack tip may be obtained as follows:

$$\sigma_{xy}^b(r,\theta) = \frac{C_b k_p}{\sqrt{(2ra)}}\frac{2Z}{h}\frac{1}{4[2+(v_1-c^2)/c^2]}\left[\left(8+\frac{3(v_1-c^2)}{c^2}\right)\cos\frac{\theta}{2}+\frac{v_1-c^2}{c^2}\cos\frac{5\theta}{2}\right]+O(r^{\frac{1}{2}}),$$

$$\sigma_{xx}^b(r,\theta) = \frac{C_b k_p}{\sqrt{(2ra)}}\frac{2Z}{h}\frac{c}{4[2+(v_1-c^2)/c^2][1-\sqrt{(v_1v_2)}]}$$

$$\times\left\{\left[8(1-v_2c^2)-\frac{v_1-c^2}{c^2}(1+7v_2c^2)\right]\sin\frac{\theta}{2}\right.$$

$$\left.+\frac{v_1-c^2}{c^2}(1-v_2c^2)\sin\frac{5\theta}{2}\right\}+O(r^{\frac{1}{2}}),$$

$$\sigma_{yy}^b(r,\theta) = \frac{C_b k_p}{\sqrt{(2ra)}}\frac{2Z}{h}\frac{c^2-v_1}{4[2+(v_1-c^2)/c^2][1-\sqrt{(v_1v_2)}]c^3}$$

$$\times\left[\left(\frac{v_1-c^2}{c^2}+8c^2-8\right)\sin\frac{\theta}{2}-\frac{v_1-c^2}{c^2}\sin\frac{5\theta}{2}\right]+O(r^{\frac{1}{2}}), \tag{50a–c}$$

where the bending component of the stress intensity ratio is evaluated in [17] to be:

$$C_b = -\frac{\{3[1-\sqrt{(\nu_1\nu_2)}]\}^{\frac{1}{2}}}{1+\sqrt{(\nu_1\nu_2)}}\sum_1^{\infty}\left(2+\frac{\nu_1-c^2}{c^2}\right)b_n. \tag{51}$$

For the isotropic shell, the foregoing analysis remains valid with $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu$, $G_{12} = G$, and $c = (E_1/E_2)^{\frac{1}{4}} = 1$. Note that the θ-dependence of the stresses in the asymptotic expressions given by (45), (48), and (50) is identical to that obtained from a flat plate by using the plane stress and a fourth order linear bending theory.

5. NUMERICAL RESULTS

The elastic constants of the orthotropic shells which are considered as examples are shown in Table 1. The table also gives the "average shear modulus" calculated from

$$G_{av} = (E_1E_2)^{\frac{1}{2}}/2[1+(\nu_1\nu_2)^{\frac{1}{2}}]. \tag{52}$$

E_1 and E_2 are the elastic moduli in the axial and in the circumferential direction, respectively.

TABLE 1. ELASTIC CONSTANTS OF THE MATERIALS

	Titanium	Graphite
E_1 (psi)	$1{\cdot}507 \times 10^7$	$1{\cdot}5 \times 10^6$
E_2 (psi)	$2{\cdot}08 \times 10^7$	40×10^6
ν_1	0·1966	0·0075
ν_2	0·2714	0·2000
G_{12}	$6{\cdot}780 \times 10^6$	$4{\cdot}0 \times 10^6$
$G_{av.}$	$7{\cdot}15 \times 10^6$	$3{\cdot}73 \times 10^6$

From the values given in the table it is seen that the measured values of the shear modulus, G_{12}, are sufficiently close to that calculated from (52) so that the materials may be considered as specially orthotropic. For these two materials and for an isotropic material with a Poisson's ratio of 1/3, the numerical results are shown in Figs. 2 and 3, and in Table 2. In all these calculations it is assumed that the cylinder is under torsion, that is, the only external load is $N_{xy} = N_0 =$ constant applied to the shell away from the crack region.

Figure 2 shows the membrane and bending components of the stress intensity factor ratios

$$C_m = k_s^m/k_p, \qquad C_b = k_s^b/k_p, \qquad k_p = N_0(\sqrt{a})/h, \tag{53}$$

as a function of the dimensionless variable $a/(Rh)^{\frac{1}{2}}$ for a mildly orthotropic material, titanium, and for an isotropic material. Since the conventional shell parameter

$$\lambda = [12(1-\nu_1\nu_2)E_2/E_1]^{\frac{1}{4}}a/(Rh)^{\frac{1}{2}}$$

is dependent on the elastic constants, in order to compare the stress intensity factors in isotropic and in orthotropic shells with the same geometry, here in presenting results $a/(Rh)^{\frac{1}{2}}$ is used as the independent variable. The figure indicates that, even for the mildly orthotropic material under consideration, the deviation of the stress intensity ratios from

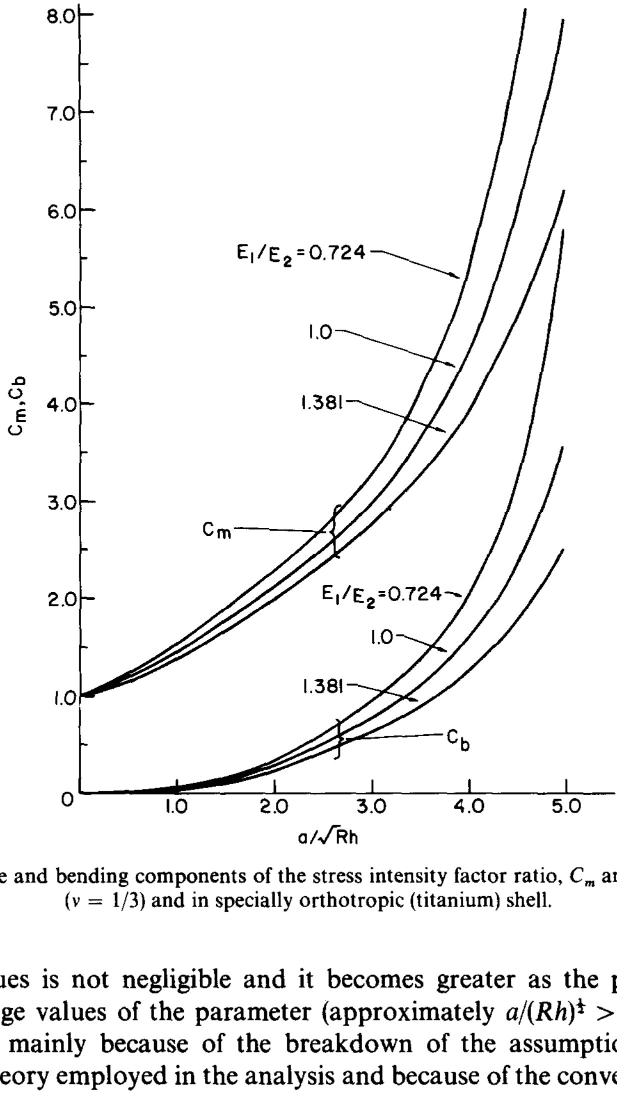

FIG. 2. Membrane and bending components of the stress intensity factor ratio, C_m and C_b in isotropic ($\nu = 1/3$) and in specially orthotropic (titanium) shell.

the isotropic values is not negligible and it becomes greater as the parameter $a/(Rh)^{\frac{1}{2}}$ increases. For large values of the parameter (approximately $a/(Rh)^{\frac{1}{2}} > 4$) the results are not very reliable mainly because of the breakdown of the assumptions regarding the particular shell theory employed in the analysis and because of the convergence difficulties encountered in the numerical calculations.

Noting that E_1 and E_2 are, respectively, the elastic moduli in the axial and in the circumferential direction, it is seen that for the same value of $a/(Rh)^{\frac{1}{2}}$ the stress intensity ratios C_m and C_b increase for decreasing E_1/E_2, that is, when the shell becomes stiffer in circumferential direction (see also Table 2).† However, this does not necessarily mean a reduction in the resistance of the shell to shear fracture. For this one also has to consider the shear fracture strength of the material in the plane parallel to E_1 as a function of E_1/E_2. It is reasonable to expect that this strength too would increase as E_1/E_2 decreases.

Figure 3 shows the stress intensity ratios C_m and C_b in an isotropic shell with $\nu = 1/3$. Here the independent variable is the conventional shell parameter $\lambda = [12(1-\nu^2)]^{\frac{1}{4}}a/(Rh)^{\frac{1}{2}}$

† This is, of course, primarily due to the multiplicative factor $(E_2/E_1)^{\frac{1}{4}}$ in the expression of λ. In fact for a quick estimate of C_m and C_b in a specially orthotropic shell, the results given in Fig. 3 are sufficient provided λ is calculated for the orthotropic shell.

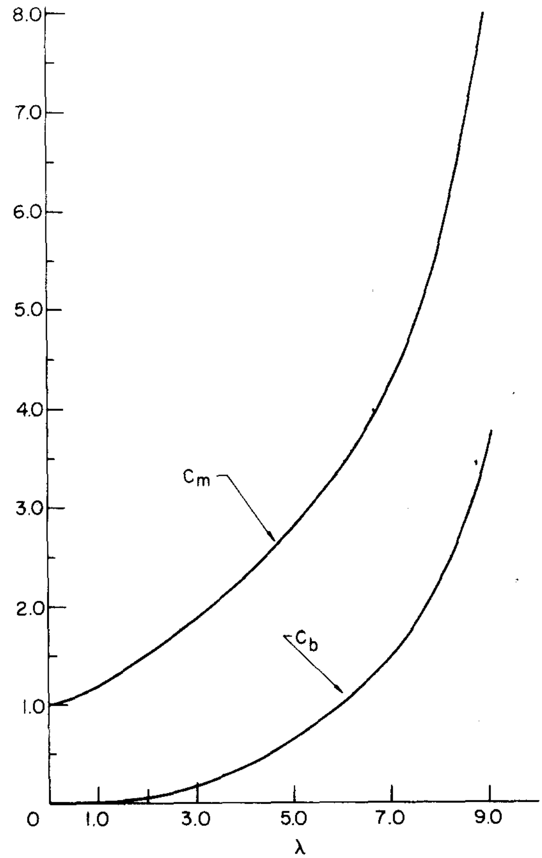

FIG. 3. Stress intensity factor ratios, C_m, C_b, vs. the shell parameter λ ($\nu = 1/3$), in isotropic shells.

which was used in the previous investigations. For a fixed value of $a/(Rh)^{\frac{1}{2}} = 1{\cdot}660$, the results found for an isotropic material (with $\nu = 1/3$), for titanium, and for graphite are shown in Table 2. These results indicate that as the degree of anisotropy increases, for the same geometry the deviation from the isotropic results also increases.

TABLE 2. THE EFFECT OF ORTHOTROPY ON THE STRESS INTENSITY RATIOS ($a/(Rh)^{\frac{1}{2}} = 1{\cdot}660$)

	Isotropic material	Titanium		Graphite	
E_1/E_2	1·0	1·381	0·724	26·667	0·0375
λ	3·0	2·811	3·304	1·359	7·018
C_m	1·942	1·880	2·044	1·340	4·045
C_b	0·199	0·158	0·239	0·0187	1·241

In the shells the Poisson's ratio appears in the analysis independently as well as through the parameter λ. In the previous studies [1–9] the results were given for $\nu = 1/3$ only. In order to have some idea about the effect of ν in the isotropic shells, the stress intensity ratios, C_m and C_b were calculated in the range $0 \leq \nu \leq 0{\cdot}5$ for a fixed value of $a/(Rh)^{\frac{1}{2}} = 2{\cdot}686$ (which corresponds to $\lambda = 5$ at $\nu = 1/3$). The results are shown in Fig. 4. Even though this

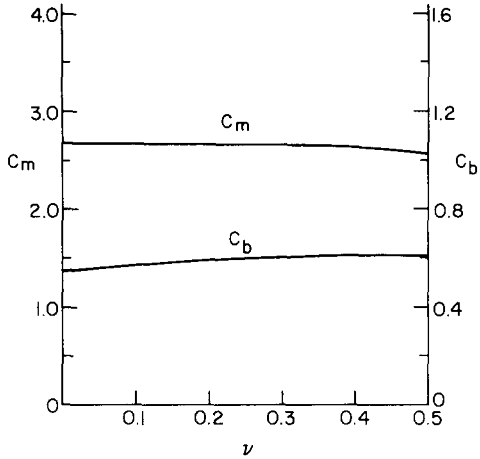

FIG. 4. The effect of Poisson's ratio on the stress intensity factor ratios in an isotropic shell ($a/(Rh)^{\frac{1}{2}} = 2{\cdot}686$).

is only for one value of $a/(Rh)^{\frac{1}{2}}$, it may, nevertheless, be concluded that in isotropic shells the effect of the Poisson's ratio on the stress intensity factors is not expected to be significant.

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APPENDIX A

The kernels $h_{ij}(x, t, y)$, $(i, j = 1, 2)$:

$$h_{1j}(x, t, y) = \int_0^\infty F_{1j}(\alpha, y) \sin \alpha(t-x)\, \mathrm{d}\alpha, \qquad (j = 1, 2), \tag{A.1}$$

$$F_{11}(\alpha, y) = \frac{1}{2}\left[-\frac{\alpha^2}{i_1\lambda}n_1 - \alpha n_2 - \frac{\alpha^2}{i_2\lambda}n_3 - \alpha n_4\right], \tag{A.2}$$

$$F_{12}(\alpha, y) = \frac{1}{2}\left[-i_1\lambda n_1 - \alpha n_2 + \left(\frac{2\alpha^2(v_1 - c^2)}{i_2\lambda c^2} + i_2\lambda\right)n_3 + \left(\frac{2\alpha(v_1 - c^2)}{c^2} + \alpha\right)n_4\right], \tag{A.3}$$

$$\begin{aligned} n_1 &= \frac{e^{m_1 y}}{m_1} - \frac{e^{m_2 y}}{m_2}, \qquad n_2 = \frac{e^{m_1 y}}{m_1} + \frac{e^{m_2 y}}{m_2}, \\ n_3 &= \frac{e^{m_3 y}}{m_3} - \frac{e^{m_4 y}}{m_4}, \qquad n_4 = \frac{e^{m_3 y}}{m_3} + \frac{e^{m_4 y}}{m_4}, \end{aligned} \tag{A.4}$$

$$h_{2j}(x, t, \alpha) = \int_0^\infty F_{2j}(\alpha, y)[\sin\alpha(t-x) - \sin\alpha t]\,d\alpha, \qquad (i, j = 1, 2) \tag{A.5}$$

$$\begin{aligned} F_{21}(\alpha, y) = \frac{1}{2}\Bigg[&\left(c^2 - v_1 - \frac{h^3 G_{12}}{3D_2}\right)\left(\frac{\alpha^2}{i_1\lambda}n_1 - \frac{\alpha^2}{i_2\lambda}n_3\right) \\ &+ \left(2c^2 - v_1 - \frac{h^3 G_{12}}{3D_2}\right)\alpha(n_2 - n_4) + i_1\lambda c^2 n_1 - i_2\lambda c^2 n_3\Bigg], \end{aligned} \tag{A.6}$$

$$\begin{aligned} F_{22}(\alpha, y) = \frac{1}{2}\Bigg[&\left(c^2 - v_1 - \frac{h^3 G_{12}}{3D_2}\right)\left(\alpha n_2 + \frac{2\alpha^2(v_1 - c^2)}{i_2\lambda c^2}n_3 + \alpha n_4\right) \\ &+ \left(2c^2 - v_1 - \frac{h^3 G_{12}}{3D_2}\right)\left(i_1\lambda n_1 + \frac{2\alpha(v_1 - c^2)}{c^2}n_4 + i_2\lambda n_3\right) \\ &+ \frac{i\lambda^2 c^2}{\alpha}n_2 + 2i_2\lambda(v_1 - c^2)n_3 - \frac{i\lambda^2 c^2}{\alpha}n_4\Bigg]. \end{aligned} \tag{A.7}$$

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Абстракт—Исследуется кососимметрическая задача для цилиндрической оболочки, имеющей осебую трещину. Подразумевается, что материал характеризуется специальной ортотропией, именно такой,что можно правильнее определить модуль сдвига путем измерения модулей Юнга и коэффициента Пуассона, чем представить модуль сдвига в качестве независимой постоянной материала. Задача решена в рамках линеаризованной теории пологих оболочек восьмого порядка. В качестве численных примеров, исследуются кручение изотропного цилиндра и кручение специально ортотропного цилиндра (титан). Вычисляются мембранные и изгибные компоненты фактора интенсивности напряжений и даются в виде функций безразмерного параметра оболочки. В задаче кручения для цилиндра с осебыми трещинами, эффекты изгиба окагываются более значительными по сравнению с задачей кручения цилиндрической оболочки с трещинами по окружности. Также, если параметр оболочки увеличивается, в отличие от результатов вычисленных для оболочек под давлением, изгибные напряжения, вокруг концов трещины, не изменяют знака.